Various Aspects of the Structural Theory of Liquids

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Liquids are defined as belonging to a new class of structures, namely zero internal momentum mode structures as distinguished from statically determinate and indeterminate structures. From the structural point of view, zero momentum modes are fundamental to liquids. Constraint equations from the zero momentum condition result in irrotational modes of displacement for both statics and dynamics. Steady flow of incompressible liquids is treated as the initial structural response due to impulsive loads, small displacements.

Introduction

THE procedure and rationale for treating liquids as an array of structural elements were introduced in previous papers.[1,2] Application was to vibration problems wherein liquid mass and stiffness matrices were formulated. The first paper considered liquid structure (container) interaction and presented a method for computing the liquid mass matrix with respect to container coordinates. Surface tension, compressibility, and gravity forces were taken into account in the second paper. Procedures were developed for computing mass and stiffness matrices which formulated the slosh and tsunami problems. Other investigators have since taken up the structural approach to the liquid container problem. In Ref. 3, the structural finite element approach is applied to simple sloshing and results compared to those from finite-difference and other finite element methods. Reference 4 formulates the structural approach of Refs. 1 and 2 in an alternative manner using an explicit displacement function and solves a simple slosh problem with indication of improved accuracy of calculated slosh frequencies. Reference 5 uses a similar formulation as Ref. 4, develops a liquid tetrahedron element, and reports development of a computer code for calculation of liquid structure interaction vibration modes and frequencies.

The structural approach to the problem of a liquid surrounded by an elastic container introduced in Refs. 1 and 2 is applicable to biomechanics. The lumbar intervertebral disk is treated in such fashion in Ref. 6. Static stresses due to external forces acting on an idealized elastic container are obtained. In addition, a doctoral thesis in preparation at the Univ. of California at Los Angeles is employing the theory as applied to idealizing the brain and other organs for structural dynamic response resulting from accidents.[7]

Although the major application of the liquid structure approach may continue to be to liquid structure interaction, further developments are possible as applied to other problems. In the present paper, fundamental structural properties of liquids are pursued without reference to immediate technical application.

Zero momentum internal circulation modes which are fundamental to liquids were briefly discussed in Refs. 1 and 2. Zero momentum modes have been discussed in the technical literature of structural dynamics, but except for rigid body modes only textbook problems have been considered, as in Ref. 8. Liquids furnish a technically important class of structures with zero momentum modes as distinguished from rigid body modes.

The present paper includes further discussion of the physical significance of the zero momentum internal circulation modes in connection with statics and dynamics of liquids. Extension of the structural theory to steady flow is developed. Liquids are a perfect plastic material and in the Appendix their plastic properties are shown on the same diagram with those for solids.

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Basic Structural Properties of Liquids

In the structural approach the liquid continuum is considered as an array of discrete structural elements. Equilibrium of an element with respect to internal pressure is by forces acting at nodes in the usual manner of discrete element structural theory with the following difference.

The liquid is assumed frictionless, and shear forces are not generated; hence, forces at a node are in the direction of a single axis and each node has only one mass degree of freedom. Therefore, several nodes are needed to represent the total forces acting on a discrete volume of liquid and to represent its mass.

This property leads to the basic structural characteristic of a liquid structure, namely the existence of zero momentum modes of displacement in dynamics and the necessity for considering zero momentum displacement modes in order to obtain a static solution. These zero momentum modes are also zero strain energy and zero frequency modes.

It is natural to inquire as to the static solution of an array of liquid elements, that is, to determine the displacement of all other nodes, given the displacements of boundary nodes. (Solution procedures for such problems may be of interest involving liquid experiments in a zero "g" environment.) It will be seen that there are not sufficient equations to solve for the unknown displacements unless zero momentum modes are considered.

In the present paper, simple two-dimensional discrete element models consisting of only a few liquid elements are discussed. This enables solution by hand calculation. The intention is to illustrate the application of structural principles to liquids, and the simple models are adequate for this purpose. Application to practical technical problems would call for detailed models and computer solution.

Static Theory of Liquid Structures

The Saint-Venant principle is useful for comparison of the liquid structure to conventional structures. It does not hold for statically determinate structures. For the static analysis of liquid structures due to external forces, stresses are everywhere constant. The concept of Saint-Venant principle is meaningless in this case. This suggests that liquid structures and continuum statically indeterminate structures are opposite types, at extremes from statically determinate structures. Liquids belong to another class of structures, namely zero internal momentum mode structures. The zero momentum mode property is now discussed in detail.

Consider the compressible liquid structure in a rigid container shown in Fig. 1. Static solution for the six displacements with the model subjected to the compression forces $P_1 = P_2 = P$ are determined. Six equations are needed to solve for the six unknown displacements.

The equation $y_1 = y_2$ can be arbitrarily assumed. For equally sized elements, if a is proportional to the volume change of an element, four equations can be written from the volume change of each element as a function of node displacements as follows:

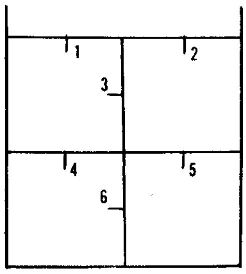

Fig. 1 Nodes of array in rigid container.

$$\begin{aligned} a &= y_1+y_3-y_5 \\ a &= y_2-y_3-y_5 \\ a &= y_4+y_6 \\ a &= y_5-y_6 \end{aligned} \tag{1}$$

(Knowing the applied force P, the liquid bulk modulus and element dimensions, the value of a can be readily computed.) Another equation is needed to solve for the six displacements, and for this equation it is necessary to consider momentum of the liquid structure.

The internal circulation mode of zero strain energy is $y_1 = y_2 = 0$, $-y_4 = y$, $-y_3 = y_5 = y_6 = y$. This can be written in matrix form as

$$\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \quad y = Fy \tag{2}$$

There is no possibility of motion in this displacement mode as a result of the external forces. Thus the momentum in this mode must be zero. That is,

$$F^t M q = 0$$

The F matrix is defined above, M is the mass matrix, and q the structural coordinates. This expression is derived in a later section.

The following mass matrix is assumed

$$\begin{bmatrix} 0.5 & & & & & \\ & 0.5 & & & & \\ & & 1.0 & & & \\ & & & 2.0 & & \\ & & & & 1.0 & \\ & & & & & 2.0 \end{bmatrix}$$

This represents a liquid model with a heavier liquid in the lower left bottom element than in the other three. This asymmetrical weight distribution is arbitrarily chosen, as otherwise a relation among coordinates could be obtained from symmetry considerations without bringing out the fundamental momentum concept.

The equation $F^t M q = 0$ results in the equation

$$-y_3+y_5+2y_6-2y_4 = 0 \tag{3}$$

The solution for nodal displacements are obtained from the six equations and are $y_1 = y_2 = 2a$, $y_3 = -a/6$, $y_4 = 5a/6$, $y_5 = 7a/6$, $y_6 = a/6$.

We see that for static analysis of liquid structures, it is necessary to consider the mass matrix of the structure. The resulting solution is an irrotational displacement mode which parallels irrotational flow of conventional fluid dynamics.

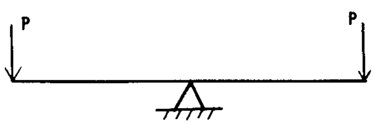

Fig. 2 Beam with liquid properties.

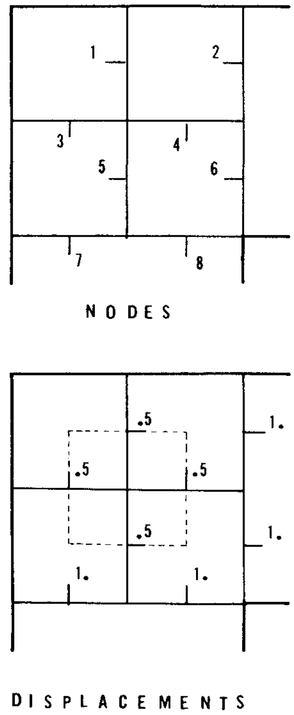

Fig. 3 Nodes and displacements of array with elbow boundary.

The beam loaded and supported as shown in Fig. 2 also has liquid structural properties involving a force relation for static equilibrium and a zero momentum mode. Thus these properties are not unique to liquids, and other structures can be classed as zero internal momentum mode structures.

Consider the simple array of incompressible liquid elements in an elbow type container as shown in Fig. 3. Given the boundary static displacements $y_6 = y_7 = y_8 = -1.0$, the problem is to determine the remaining nodal displacements. Five equations are needed to solve for the five remaining unknown nodal displacements. Four equations can be written from continuity as follows:

$$\begin{aligned} y_1 &= y_3 \\ y_2 &= y_1+y_4 \\ y_7 &= y_3+y_5 \\ y_5+y_8 &= y_4+y_6 \end{aligned} \tag{4}$$

Another equation is needed to solve for the five unknowns, and again it is necessary to consider the momentum of the liquid structure.

The internal circulation mode of zero strain energy F is defined by

$$\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{Bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad y = Fy \tag{5}$$

There is no possibility of motion in this mode because of the assumed boundary displacements. Hence, momentum in the mode is zero. Then for an assumed mass matrix consisting of a unit matrix of order 8, the expression $F^t M q = 0$ results in the additional equation

$$-y_1+y_4+y_5-y_3 = 0 \tag{6}$$

The nodal displacements are determined by solving the five equations represented by Eqs. (4) and (6). The final displacements are shown in Fig. 3.

Equation (6) can be derived in another manner. Considering only nodes involved in the internal circulation mode, trace a

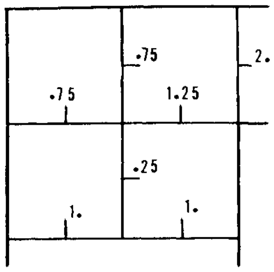

Fig. 4 Displacements of array with restricted exit.

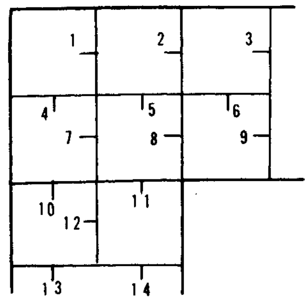

Fig. 6 Nodes of array with elbow boundary.

path consecutively from one node to the next as indicated by the dotted line in Fig. 3. Taking the nodal displacement unknowns as positive, if their positive direction is in the same direction as the path traversed, Eq. (6) is the sum of the products of the nodal mass and displacement unknowns. Note that nodal masses in the example are equal. This technique parallels calculation of circulation about a closed path from fluid dynamics theory.

The static displacements for two incompressible liquid arrays shown in Figs. 4 and 5 are obtained in the same manner as for Fig. 3. All but one boundary displacement are given displacements. (Continuity would be violated if all boundary displacements were independent, that is, given.) The remaining number of unknown displacements is equal to the sum of the number of continuity and zero momentum mode equations. Thus these equations are just sufficient for determination of the unknown displacements.

On examination of the displacement solutions of Figs. 4 and 5, it is seen that continuity is satisfied. Computing the momentum of the internal circulation modes as a circulation calculation, as previously described for the example of Fig. 3, it is seen that zero momentum is also satisfied. Moreover, these displacement values are unique. Only the values shown satisfy continuity and zero momentum for the given boundary displacements.

Matrix Formulation for Eliminating Zero Momentum Modes

If F is the matrix of zero momentum modes, generalized structural coordinates q in terms of zero momentum mode coordinates p are

$$q = Fp \tag{7}$$

External forces do not cause displacements in the F modes and the strain energy due to motion in the F modes is zero. The F matrix is readily formed in terms of generalized coordinates as shown in examples discussed in the next section. In each mode described by the F matrix, all elements of a column are zero except for one corresponding to an internal generalized coordinate. Although at first sight it does not appear so, the F matrix is easily formed in this manner.

The momentum with respect to the q coordinates is

$$M\dot{q}$$

Momentum is a vector. Then the momentum with respect to the p coordinates is

$$F^t M\dot{q}$$

As momentum in the p modes is zero

$$F^t M\dot{q} = 0 \quad \text{and} \quad F^t Mq = 0 \tag{8}$$

These are the constraint equations for eliminating the zero momentum modes. The result is irrotational motion which parallels irrotational flow of conventional fluid dynamics. Irrotational flow is very much a constraint type of motion from the structural point of view.

Dynamic Theory of Liquid Structures

The static displacement solutions of Figs. 3–5 can be regarded as steady flow solutions for an incompressible liquid. The displacements shown in the figures are regarded as velocities. This is reasoned by first considering these velocities as the structural dynamic response due to impulsive loads at the boundary. The velocities are independent of time as the stiffness matrix is zero. Then the initial solution just after application of the impulsive loads represents the steady flow solutions.

A matrix solution for displacements (velocities) as a function of assumed boundary displacements (velocities) is now given with reference to the array of incompressible liquid elements shown in Fig. 6. The structural coordinates $y(y_1 \ldots y_{14})$ are related to generalized coordinates $q(y_3, y_7, y_8, y_{12}, y_{13}, y_{14})$ by a T matrix from continuity considerations as

$$y = Tq \tag{9}$$

If M_y is the mass matrix with respect to the y coordinates, the mass matrix with respect to the q coordinates is

$$M_q = T^t M_y T \tag{10}$$

The zero momentum mode matrix F in the q coordinate system is

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

These modes cannot be excited by forces at the boundaries. They can only be excited by forces which act at interior nodes and these forces are zero.

The constraint equations are formed from the condition of zero momentum in these modes or

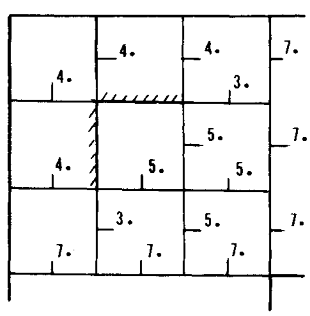

Fig. 5 Displacements of array with baffle.

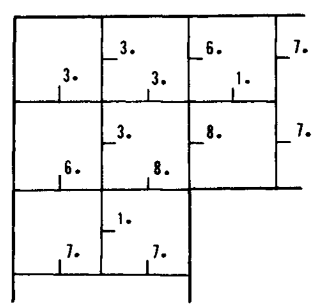

Fig. 7 Velocities of array shown in Fig. 6.

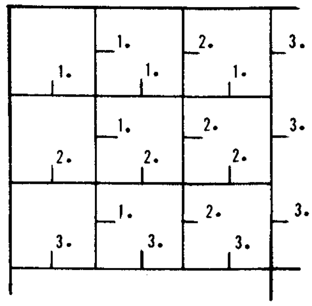

Fig. 8 Velocities.

$$F^t M_q q = 0 \tag{11}$$

This system of equations can be written as

$$A\begin{Bmatrix} y_7 \\ y_8 \\ y_{12} \end{Bmatrix} + B\begin{Bmatrix} y_3 \\ y_{13} \\ y_{14} \end{Bmatrix} = 0 \tag{12}$$

Then

$$\begin{Bmatrix} y_7 \\ y_8 \\ y_{12} \end{Bmatrix} = -A^{-1}\ B\begin{Bmatrix} y_3 \\ y_{13} \\ y_{14} \end{Bmatrix} \tag{13}$$

Coordinates y_7, y_8, and y_{12} are determined for any assumed boundary values for y_3, y_{13}, and y_{14} from Eq. (13). The remaining coordinates are determined from Eq. (9). Note that all boundary values except one are given. As previously mentioned, continuity could be violated if all boundary coordinates are independent. A numerical example for $y_3 = y_{13} = y_{14} = -7.$ is shown in Fig. 7.

The velocities shown in Fig. 8 for another array are obtained in a similar manner. The principle of linear momentum (the rate of change of linear momentum in any fixed direction is equal to the resolved force in that direction) is applied in orthogonal directions separately to obtain the resulting differential element pressures shown in Fig. 9. The resulting total pressure based on the differential pressure is shown in Fig. 10. Note that Figs. 8–10 refer to the same array of liquid elements.

The velocity distribution for symmetrical two-dimensional irrotational incompressible flow around a point can be derived from the zero momentum condition. Consider the differential discrete liquid elements of Fig. 11. There is no flow in the radial direction. Liquid velocities in the adjacent elements are v and $v+dv$ and their masses are proportional to r and $r+dr$, respectively. The zero momentum condition gives

$$(v+dv)(r+dr) - vr = 0 \tag{14}$$

which reduces to

$$dv/v = -dr/r \tag{15}$$

The differential equation has the solution $v = k/r$ which is the known solution for a potential vortex.

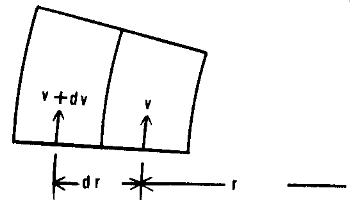

Fig. 11 Potential vortex motion idealization.

Sources and sinks, consisting only of radial flow, satisfy the zero momentum condition. Their velocities are computed from the continuity condition.

The effect of friction on the flow of an incompressible liquid is discussed by considering Fig. 12. A solution for the same array but with a frictionless liquid is shown in Fig. 5. (All liquid arrays noted so far assumed a frictionless liquid.) In Fig. 12, it will be assumed that velocities at nodes 2 and 6 are equal because of a friction shear force generated at the discontinuity of velocities between nodes 2 and 6 as indicated by Fig. 5.

With reference to Fig. 12, generalized coordinates q can be taken as $y_2, y_6, y_7, y_{13}, y_{14}, y_{15}$, and y_{16}. The total coordinates $y(y_1 \ldots y_{16})$ are related to the generalized coordinates by Eq. (9). The T matrix is generated from continuity considerations and is given below for square liquid elements

$$\begin{bmatrix} 1 & & & & & & \\ 1 & & & & & & \\ & & -1 & -1 & 1 & 1 & 1 \\ 1 & & & & & & \\ -1 & & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & & & & & \\ & & 1 & & & & \\ & 1 & & & & & \\ -1 & -1 & & -1 & 1 & 1 & 1 \\ -1 & & & & 1 & & \\ -1 & -1 & & & 1 & 1 & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix}$$

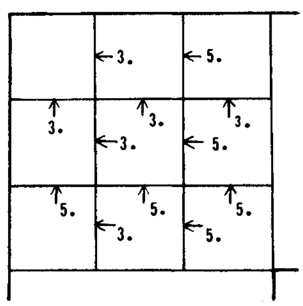

Fig. 9 Differential element pressure.

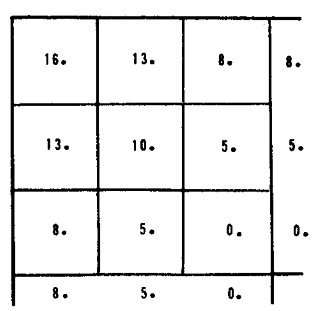

Fig. 10 Pressure distribution in an array of elements.

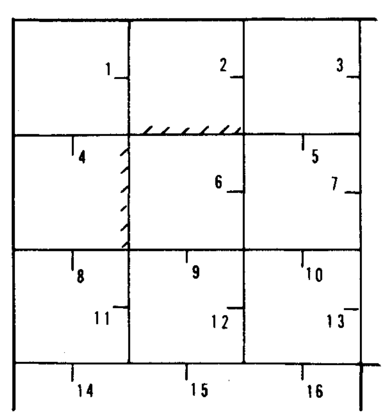

Fig. 12 Nodes of array with baffle.

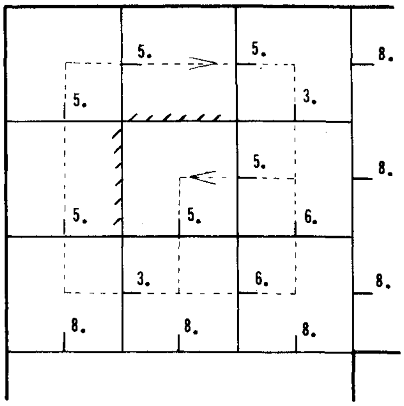

Fig. 13 Velocities with friction forces.

The mass matrix with respect to the q coordinates M_q is computed from Eq. (10). If the M_y mass matrix is a unit matrix of order 16, the M_q matrix is the following:

$$\begin{bmatrix} 8 & 2 & 1 & 2 & -4 & -3 & -2 \\ 2 & 4 & 0 & 1 & -2 & -2 & -1 \\ 1 & 0 & 3 & 2 & -2 & -2 & -2 \\ 2 & 1 & 2 & 4 & -3 & -3 & -3 \\ -4 & -2 & -2 & -3 & 6 & 4 & 3 \\ -3 & -2 & -2 & 3 & 4 & 5 & 3 \\ -2 & -1 & 2 & -3 & 3 & 3 & 4 \end{bmatrix}$$

Momentum in the two internal circulation modes F_1 and F_2 where $F_1{}^t = [-1000000]$ and $F_2{}^t = [0100000]$ must be equal, as the same inernal friction force acts on each mode. Then from Eq. (8)

$$F_1{}^t M_q q = F_2{}^t M_q q \tag{16}$$

From the equation $y_2 = y_6$ and Eq. (16), for the boundary values $y_7 = y_{13} = y_{14} = y_{15} = y_{16} = -8$, the values $y_2 = y_6 = -5$ are obtained. With these values, the complete solution is calculated from Eq. (9), and it is shown in Fig. 13.

Two internal circulation modes are indicated in Fig. 13. Momentum in each mode is computed as the sum of the products of nodal velocity and nodal mass while tracing a path from one node to the next as indicated by the dotted lines. Velocities are taken positive if their computed direction (Fig. 13) is in the same direction as the path traversed. In this manner, it is shown that the total momentum in each mode is equal. This discussion of friction forces, and resulting motion of liquid arrays based on structural dynamic concepts, brings out several points explained by circulation theory as applied to aerofoils.[9]

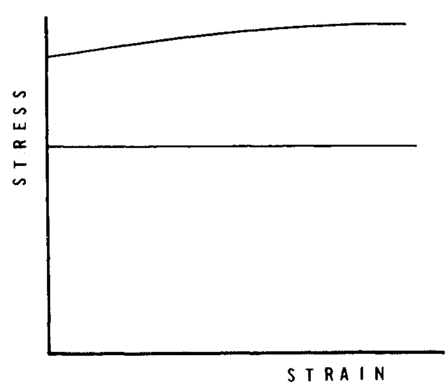

Fig. 14 Plastic effective stress and effective strain.

Conclusions

Liquids can be idealized as structures by the discrete element technique. It is shown that the concept of modes and zero momentum modes, and the technique of elimination of modes, all from well-known structural dynamic theory, can be applied to liquids. In this way, a structural interpretation can be given to irrotational motion, circulation along a closed path and circulation theory of aerofoils.

In the present paper, these concepts are explained by considering simple liquid structural arrays of elements without going into applications. However, since the introduction of the liquid structural concept, application has been made by others in the study of liquid structure interaction. Further developments in interdisciplinary areas could extend applications beyond these modest beginnings.

Appendix

In plastic theory of solids, a universal relation is assumed to exist between effective stress and effective plastic strain for a particular solid.[10] A typical relation for a solid with work hardening properties is shown by the curve in Fig. 14. The horizontal relation is for a perfect plastic material. The effective stress in a liquid is always zero and therefore the plastic relation for liquids is a line coincident with the abcissa of Fig. 14.

A liquid contained by a long trough is in the same condition regarding the propagation of a stress wave imparted by a movable end as is a rod with a stress wave superimposed on an initial plastic stress state. In a liquid, an initial disturbance would travel in such a trough at an elastic stress wave velocity based on the bulk modulus.[11] Just as for an initially stressed rod, the major portion of the disturbance in the liquid would travel at a much lower velocity.

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